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## Optimized $\delta$ expansion for relativistic nuclear models

G. Krein<sup>\*a</sup>, R.S. Marques de Carvalho<sup>b</sup>, D.P. Menezes<sup>b</sup>, M. Nielsen<sup>c</sup>, M.B. Pinto<sup>b</sup>

<sup>a</sup> *Institut für Kernphysik, Universität Mainz, D-55099 Mainz, Germany*

*and*

*Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145,  
01405-900 São Paulo, S.P. Brazil*

<sup>b</sup> *Departamento de Física, Universidade Federal de Santa Catarina  
88.040-900 Florianópolis, S.C., Brazil*

<sup>c</sup> *Instituto de Física, Universidade de São Paulo, Caixa Postal 66318  
5389-970 São Paulo, S.P., Brazil*

### Abstract

The optimized  $\delta$ -expansion is a nonperturbative approach for field theoretic models which combines the techniques of perturbation theory and the variational principle. This technique is discussed in the  $\lambda\phi^4$  model and then implemented in the Walecka model for the equation of state of nuclear matter. The results obtained with the  $\delta$  expansion are compared with those obtained with the traditional mean field, relativistic Hartree and Hartree-Fock approximations.

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\* Alexander von Humboldt research fellow.

## 1. Introduction

The study of possible modifications of hadron properties in the nuclear medium is one of the central problems of contemporary nuclear physics. In principle, these and related phenomena in nuclear physics are governed by the fundamental theory of the strong interactions, quantum chromodynamics (QCD). However, although QCD has been very successful in explaining a large class of hadronic processes at high energy and large momentum transfer, typical nuclear phenomena at lower energies cannot be derived from QCD with the theoretical tools presently available. The difficulty of using QCD for phenomena at the nuclear scale is related to the nonperturbative nature of these. Due to the asymptotic freedom property of QCD, high energy processes are calculable by perturbative techniques in the quark-gluon coupling constant. On the other hand, since there are no reliable systematic approximation schemes in field theory for performing nonperturbative calculations, the construction of models is an important aspect of low energy QCD. While there is considerable optimism that eventually one will be able to solve QCD numerically on the lattice using supercomputers, the development of analytical approximation methods are in urgent need to make contact with the wealth of data on nonperturbative phenomena presently available, or that will be available when the new experimental facilities under construction start operating. The  $\delta$  expansion [1] is an example of a method recently developed aiming to study nonperturbative phenomena in field theory.

The idea of the  $\delta$  expansion is to perturb the original theory by the introduction of an artificial expansion parameter  $\delta$ , absent in the original theory. The parameter  $\delta$  is introduced in such a way that it interpolates between the theory one wants to solve and another theory that one knows how to solve. The  $\delta$  expansion can be formulated in two different forms, the logarithmic  $\delta$  expansion [1] and the linear  $\delta$  expansion [2]- [4]. In this paper we consider the linear form. Specifically, let  $\mathcal{L}$  be the Lagrangian density of the theory one wants to solve, and  $\mathcal{L}_0$  the Lagrangian density of the soluble theory. Then, the interpolating Lagrangian density  $\mathcal{L}(\delta)$  is defined as

$$\mathcal{L}(\delta) = (1 - \delta)\mathcal{L}_0 + \delta \mathcal{L} = \mathcal{L}_0 + \delta (\mathcal{L} - \mathcal{L}_0), \quad (1)$$

so that  $\mathcal{L}(0) = \mathcal{L}_0$ ,  $\mathcal{L}(1) = \mathcal{L}$  and  $\mathcal{L}_0$  is a function of an arbitrary mass parameter  $\mu$ . The next step involves the evaluation of desired physical quantities as a perturbation series in

powers of  $\delta$ , and then  $\delta$  is set equal to 1 at the end. A crucial aspect of the method is the recognition that  $\mathcal{L}_0$  involves arbitrary unknown (dimensionful and/or dimensionless) parameters.

Fixing the arbitrary parameter  $\mu$  is the step which brings all nonperturbative information contained in the perturbative calculation. Several ways to fix the arbitrary mass parameter have been proposed in both versions of the  $\delta$  expansion as well as in the related methods. One physically appealing way to fix the unknown parameters, which is the one adopted here, is the principle of minimal sensitivity (PMS) introduced in Ref. [5]. This variational principle amounts to the requirement that a physical quantity  $P(\mu)$  should be at least locally independent of these parameters, which implies that

$$\frac{\partial P(\mu)}{\partial \mu} \Big|_{\bar{\mu}} = 0, \quad (2)$$

at  $\delta = 1$ . The solution to the PMS equation gives  $\bar{\mu}$  as a function of the original parameters of the theory including the coupling constant. The  $\delta$  expansion, together with the criterion of the PMS of physical observables, is known as the *optimized  $\delta$  expansion*. The convergence of the optimized  $\delta$  expansion has been proved in Ref. [7] for a quantum mechanical problem.

The different forms of the  $\delta$  expansion have been successfully applied to many different problems in quantum mechanics [6], particle theory [8]- [10], statistical physics [11] and lattice field theory [3]- [12]. Most applications show that one is always able to reproduce traditional non perturbative results already at lowest order in  $\delta$ .

Obviously different approximations give different prescriptions so as to select a subset of Feynman diagrams among the infinite set which describes a physical process and within the  $\delta$  expansion this selection is done in an essentially perturbative way. Also, as we shall explicitly see, the same order in  $\delta$  can contain diagrams which would belong to different orders if we were using other approximations. A drawback of traditional nonperturbative analytical approximations is that one has to sum an infinite subset of graphs so as to consider all orders in the coupling. This procedure generates problems related to the inclusion of higher orders or nonperturbative renormalization or both.

One advantage of the method presented here is that one deals with a reduced number of Feynman graphs so that renormalization can be carried out in a perturbative way before the PMS produces the final finite nonperturbative results. Also, because there is no self

consistency involved, it can be considered more economical as far as numerical computations are concerned. Motivated by these advantages, we have recently [13] implemented the optimized  $\delta$  expansion for the Walecka model [14]. We have investigated vacuum effects by neglecting exchange diagrams and have shown that the relativistic Hartree approximation results are exactly reproduced. In a forthcoming work we will present results which also include exchange diagrams.

In the present work we do not address the renormalization question by ignoring vacuum effects. Here only matter effects are considered up to second order in  $\delta$  which includes direct as well as exchange graphs. The results are compared with the traditional Hartree and Hartree-Fock approximations. Our aim is just to establish the reliability of the method in coping with nuclear matter problems. As a byproduct we hope to provide the reader with a powerful alternative tool which can be used in investigations aiming to include higher order contributions (such as vertex corrections) and vacuum effects.

Before launching into the actual applications a last remark on how to implement the optimized  $\delta$  expansion is in order. The standard procedure is to expand the physical quantity of interest ( $P$ ) in orders of  $\delta$  starting with the interpolated Lagrangian density  $\mathcal{L}(\delta)$ . For example if  $P$  is the energy density ( $\mathcal{E}$ ) one calculates vacuum to vacuum diagrams order by order as in perturbation theory using the  $\mathcal{O}(\delta^0)$  propagator. In this way  $\delta$  labels the diagrams contributing to  $\mathcal{E}$  and improvements will eventually result from the inclusion of higher order terms. Alternatively one can obtain an exact expression for  $\mathcal{E}$  using the energy-momentum tensor ( $T^{\mu\nu}$ ) derived from the *original* theory  $\mathcal{L}(1)$ . In this case the final expression for the energy density is obtained in terms of the full propagators, which are then evaluated via the  $\delta$  expansion.

In the present work we adopt the latter prescription for the Walecka model. However, the standard prescription is being used in a forthcoming work where the energy density is derived perturbatively from the generating functional of the interpolated theory. We will then be able to check the equivalence between both prescriptions.

The paper is organized as follows: in the next section we use the  $\lambda\phi^4$  model to pedagogically introduce the  $\delta$  - expansion method. In section 3 we apply the  $\delta$ - expansion to the Walecka model and in section 4 we present our conclusions.

## 2. The $\lambda\phi^4$ model

To start with, we consider the scalar  $\lambda\phi^4$  theory whose Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \quad (3)$$

To implement the linear  $\delta$  expansion one can consider a general free scalar Lagrangian density such as

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_0^2\phi^2, \quad (4)$$

where

$$m_0^2 \equiv m^2 + \mu^2 \quad (5)$$

$\mu$  being an arbitrary mass parameter. Then, according to Eq. (1) one gets

$$\mathcal{L}(\delta) = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}(m^2 + \mu^2)\phi^2 - \delta\left(\frac{\lambda}{4!}\phi^4 - \frac{1}{2}\mu^2\phi^2\right) \quad (6)$$

The general way the method works becomes clear by looking at the Feynman rules generated by  $\mathcal{L}(\delta)$ . First, the original  $\phi^4$  vertex has its original Feynman rule  $-i\lambda$  modified to  $-i\delta\lambda$ . This minor modification is just a reminder that one is really expanding in orders of the artificial parameter  $\delta$ . Most importantly let us look at the modifications implied by the addition of the arbitrary quadratic part. The original bare propagator

$$i\Delta(p^2) = \frac{i}{p^2 - m^2 + i\epsilon}, \quad (7)$$

becomes at zeroth order in  $\delta$

$$i\Delta(p^2) = \frac{i}{p^2 - m^2 - \mu^2 + i\epsilon} = \frac{i}{p^2 - m^2 + i\epsilon} \left[1 - \frac{(-i\mu^2)i}{p^2 - m^2 + i\epsilon}\right]^{-1}, \quad (8)$$

or

$$\begin{aligned} i\Delta(p^2) &= \frac{i}{p^2 - m^2 + i\epsilon} + \frac{i}{p^2 - m^2 + i\epsilon} \frac{(-i\mu^2)}{p^2 - m^2 + i\epsilon} \frac{i}{p^2 - m^2 + i\epsilon} \\ &+ \frac{i}{p^2 - m^2 + i\epsilon} \frac{(-i\mu^2)}{p^2 - m^2 + i\epsilon} \frac{i}{p^2 - m^2 + i\epsilon} \frac{(-i\mu^2)}{p^2 - m^2 + i\epsilon} + \dots, \end{aligned} \quad (9)$$

indicating that the term proportional to  $\mu^2\phi^2$  contained in  $\mathcal{L}_0$  is entering the theory in a nonperturbative way. On the other hand, the piece proportional to  $\delta\mu^2\phi^2$  is only being

treated perturbatively as a quadratic vertex (of weight  $i\delta\mu^2$ ). Since only an infinite order calculation would be able to compensate for the infinite number of  $(-i\mu^2)$  insertions contained in Eq. (9) one always ends up with a  $\mu$  dependence in any quantity calculated to finite order in  $\delta$ .

Following the procedure outlined in the Introduction, the final expression for the quantity  $P$  one wants to evaluate is written in terms of the full propagators which, for the  $\lambda\phi^4$  theory, is:

$$i\Delta^*(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma(p^2) + i\epsilon}, \quad (10)$$

where  $\Sigma(p^2)$  is the self energy. The  $\delta$  expansion is then implemented via the substitution:

$$i\Delta^*(p^2) \rightarrow i\Delta^\delta(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma^\delta(p^2) + i\epsilon}, \quad (11)$$

where  $\Sigma^\delta(p^2)$  is calculated perturbatively in powers of  $\delta$ . This implies  $P = P(\mu)$  and the nonperturbative results are obtained by applying the PMS directly to this quantity, as in Eq. (2).

### 3. Walecka model

In this section we consider the Walecka model [14] for nuclear matter. The Lagrangian density of the model is given by

$$\begin{aligned} \mathcal{L}_W = \bar{\psi} [\gamma_\mu (i\partial^\mu - g_\omega V^\mu) - (M - g_\sigma \phi)] \psi + \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m_\sigma^2 \phi^2) \\ - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\omega^2 V_\mu V^\mu, \end{aligned} \quad (12)$$

where  $\psi$  represents the nucleon field operators,  $\phi$  and  $V_\mu$  are respectively the field operators of the scalar and vector mesons, and  $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ .

We are interested in the energy density of the system:

$$\mathcal{E} = \frac{1}{V} \int d^3x \left( \langle \Psi | T^{00} | \Psi \rangle - \langle \text{vac} | T^{00} | \text{vac} \rangle \right), \quad (13)$$

where  $|\Psi\rangle$  is the interacting ground-state of nuclear matter,  $|\text{vac}\rangle$  is the vacuum state (zero density), and  $T^{00}$  is the 00 component of energy-momentum tensor  $T^{\mu\nu}$ :

$$T_W^{\mu\nu} = i\bar{\psi} \gamma^\mu \partial^\nu \psi + \partial^\mu \phi \partial^\nu \phi + \partial^\nu V_\lambda F^{\lambda\mu} - g^{\mu\nu} \mathcal{L}_W. \quad (14)$$

Note that we have not used the nucleon equation of motion.

Next, we express the energy density in terms of full propagators and full self-energies [15]:

$$\begin{aligned} \mathcal{E}_W = & -i \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left\{ S(k) \left[ \gamma^0 k^0 - (\gamma^\mu k_\mu - M) \right] \right\} - i \int \frac{d^4 k}{(2\pi)^4} \Delta_\sigma(k) \left[ \frac{1}{2} (k^2 - m_\sigma^2) - (k^0)^2 \right] \\ & + i \int \frac{d^4 k}{(2\pi)^4} \Delta_{\omega\mu}^\mu(k) \left[ \frac{1}{2} (k^2 - m_\omega^2) - (k^0)^2 \right] + i \int \frac{d^4 k}{(2\pi)^4} [\Pi_\sigma(k) \Delta_\sigma(k) + \Pi_\omega^{\mu\nu}(k) \Delta_{\omega\mu\nu}(k)] \\ & - \langle \text{vac} | T^{00} | \text{vac} \rangle, \end{aligned} \quad (15)$$

where  $S(k)$ ,  $\Delta_\sigma(k)$  and  $\Delta_\omega^{\mu\nu}(k)$  are respectively the nucleon, scalar- and vector-meson full propagators, and the meson self-energies  $\Pi_\sigma(k)$  and  $\Pi_\omega^{\mu\nu}(k)$  are given by:

$$\begin{aligned} i\Pi_\sigma(k) = & -ig_\sigma \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [S(k+q) \Gamma_\sigma(p+q, q) S(q)] \\ & - (2\pi)^4 \delta^4(k) \left\{ g_\sigma \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[S(q)] \right\}^2, \end{aligned} \quad (16)$$

$$\begin{aligned} i\Pi_\omega^{\mu\nu}(k) = & +ig_\omega \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\gamma^\mu S(k+q) \Gamma_\omega^\nu(p+q, q) S(q)] \\ & - (2\pi)^4 \delta^4(k) \left\{ g_\omega \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[\gamma^\mu S(q)] \right\} \left\{ g_\omega \int \frac{d^4 q'}{(2\pi)^4} \text{Tr}[\gamma^\nu S(q')] \right\}. \end{aligned} \quad (17)$$

In these equations, the quantities  $\Gamma_i$ ,  $i = \sigma, \omega$  are the full meson-nucleon vertex functions. These, in turn, are solutions of Schwinger-Dyson equations that involve higher-order vertex functions (or scattering T-matrices). The corresponding bare vertices are given by:

$$\Gamma_\sigma = ig_\sigma, \quad (18)$$

$$\Gamma_\omega^\mu = -ig_\omega \gamma^\mu. \quad (19)$$

It is worth emphasizing that in Eq. (15) direct and exchange contributions as well as vertex corrections are included, independent of the order in  $\delta$  considered.

The strategy now is to evaluate the propagators (self-energies) and vertex functions according to the perturbative-variational scheme of the optimized  $\delta$  expansion discussed in the introduction. According to Eq. (1), to implement the  $\delta$  expansion one needs to introduce a  $\mathcal{L}_0$  such that:

$$\mathcal{L}_W(\delta) = (1 - \delta) \mathcal{L}_0 + \delta \mathcal{L}_W. \quad (20)$$

We choose:

$$\mathcal{L}_0 = \bar{\psi} (i\gamma_\mu \partial^\mu - M_0) \psi + \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m_\sigma^2 \phi^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_\omega^2 V_\mu V^\mu , \quad (21)$$

where

$$M_0 \equiv M + \mu . \quad (22)$$

The interpolated Walecka model is then given by:

$$\mathcal{L}_W(\delta) = \mathcal{L}_0 + \delta \left( -g_\omega \bar{\psi} \gamma_\mu V^\mu \psi + g_\sigma \bar{\psi} \phi \psi + \mu \bar{\psi} \psi \right) . \quad (23)$$

Notice that the  $\delta$  expansion technique could have also been applied to the meson fields explicitly. However, we will eliminate the meson propagators in terms of the nucleon propagator, using the exact Schwinger-Dyson equations for the meson propagators:

$$\Delta_\sigma(k) = \Delta_{\sigma 0}(k) + \Delta_{\sigma 0}(k) \Pi_\sigma(k) \Delta_\sigma(k) , \quad (24)$$

$$\Delta_\omega^{\mu\nu}(k) = \Delta_{\omega 0}^{\mu\nu}(k) + \Delta_{\omega 0}^{\mu\lambda}(k) \Pi_{\omega\lambda\sigma}(k) \Delta_\omega^{\sigma\nu}(k) , \quad (25)$$

where the meson self-energies are given in Eqs. (16)-(17). In this way, meson effects enter via the nucleon self-energies. This leaves us with only one unknown parameter,  $\mu$ , which will be fixed by the PMS condition applied to the energy density. As already discussed, the implementation of the method will be done via the nucleon propagator, which depends on the order in  $\delta$  considered.

Notice also that we could have eliminated the meson-nucleon interaction terms (the terms proportional to the meson self-energies in Eq. (15)) by using the *exact* nucleon Schwinger-Dyson equation. This would cancel half of the meson kinetic energies [15]. In the Appendix, we discuss an alternative way to derive the energy density [13], appropriate to calculations up to  $\mathcal{O}(\delta^2)$ , in which one eliminates from the beginning the meson field operators in favor of the nucleon ones.

At zeroth order in  $\delta$ , the nucleon self-energy, corresponding to the interpolated Lagrangian Eq. (23), is obviously zero, i.e.,  $\Sigma^{(0)} = 0$ . At this order, the single-particle energy is simply given by:

$$E(q) = E_0(q) = \left( \vec{q}^2 + M_0^2 \right)^{\frac{1}{2}} , \quad (26)$$

and the nucleon propagator is

$$S^0(q) = S_F^0(q) + S_D^0(q), \quad (27)$$

with

$$S_F^0(q) = (\gamma^\mu q_\mu + M_0) \frac{1}{q^2 - M_0^2 + i\epsilon}, \quad (28)$$

$$S_D^0(q) = (\gamma^\mu q_\mu + M_0) \frac{i\pi}{E_0(q)} \delta(q^0 - E(q)) \theta(P_F - |\vec{q}|), \quad (29)$$

the Feynman part of the propagator, Eq. (28) corresponding to the vacuum part and Eq. (29) corresponding to the medium part. In what follows we do not consider vacuum contributions.

At this zeroth-order, meson propagators  $\Delta_\sigma(k)$  and  $\Delta_\omega^{\mu\nu}(k)$  of Eqs. (24)-(25) are simply:

$$\Delta_\sigma(k) = \Delta_{\sigma 0}(k) + \Delta_{\sigma 0}(k) \Pi_\sigma(k) \Delta_{\sigma 0}(k), \quad (30)$$

$$\Delta_\omega^{\mu\nu}(k) = \Delta_{\omega 0}^{\mu\nu}(k) + \Delta_{\omega 0}^{\mu\lambda}(k) \Pi_{\omega\lambda\sigma}(k) \Delta_{\omega 0}^{\sigma\nu}(k), \quad (31)$$

with the self-energies  $\Pi_\sigma(k)$  and  $\Pi_\omega^{\mu\nu}(k)$  given by:

$$\Pi_\sigma(k) = -ig_\sigma^2 \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [S^0(k+q) S^0(q)] + i(2\pi)^4 \delta^4(k) \left\{ g_\sigma \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[S^0(q)] \right\}^2, \quad (32)$$

$$\begin{aligned} \Pi_\omega^{\mu\nu}(k) = & -ig_\omega^2 \int \frac{d^4 q}{(2\pi)^4} \text{Tr} [\gamma^\mu S^0(k+q) \gamma^\nu S^0(q)] \\ & + i(2\pi)^4 \delta^4(k) \left\{ g_\omega \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[\gamma^\mu S^0(q)] \right\} \left\{ g_\omega \int \frac{d^4 q'}{(2\pi)^4} \text{Tr}[\gamma^\nu S^0(q')] \right\}. \end{aligned} \quad (33)$$

Hence, we obtain for the zeroth order energy density

$$\mathcal{E}_W^{(0)} = \mathcal{E}_B^{(0)} + \mathcal{E}_\sigma^{(0),dir} + \mathcal{E}_\sigma^{(0),exc} + \mathcal{E}_\omega^{(0),dir} + \mathcal{E}_\omega^{(0),exc},$$

the following expressions:

$$\mathcal{E}_B^{(0)} = \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{\vec{q}^2 + MM_0}{E_0(q)}, \quad (34)$$

$$\mathcal{E}_\sigma^{(0),dir} = -\frac{1}{2} \frac{g_\sigma^2}{m_\sigma^2} \left[ \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{M_0}{E_0(q)} \right]^2 \quad (35)$$

$$\begin{aligned} \mathcal{E}_\sigma^{(0),exc} = & \frac{g_\sigma^2}{2} \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3 E_0(q)} \int_0^{P_F} \frac{d^3 k}{(2\pi)^3 E_0(k)} \Delta_\sigma([E_0(q) - E_0(k)]^2 - (\vec{q} - \vec{k})^2) \\ & \times \left[ \left( \frac{1}{2} - 1 \right) - [E_0(q) - E_0(k)]^2 \Delta_\sigma([E_0(q) - E_0(k)]^2 - (\vec{q} - \vec{k})^2) \right] \\ & \times \left[ E_0(q) E_0(k) - \vec{q} \cdot \vec{k} + M_0^2 \right], \end{aligned} \quad (36)$$

$$\mathcal{E}_\omega^{(0),dir} = \frac{1}{2} \frac{g_\omega^2}{m_\omega^2} \left[ \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \right]^2 \quad (37)$$

$$\begin{aligned} \mathcal{E}_\omega^{(0),exc} &= g_\omega^2 \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3 E_0(q)} \int_0^{P_F} \frac{d^3 k}{(2\pi)^3 E_0(k)} \Delta_\omega([E_0(q) - E_0(k)]^2 - (\vec{q} - \vec{k})^2) \\ &\quad \times \left[ \left( \frac{1}{2} - 1 \right) - [E_0(q) - E_0(k)]^2 \Delta_\omega([E_0(q) - E_0(k)]^2 - (\vec{q} - \vec{k})^2) \right] \\ &\quad \times \left[ E_0(q) E_0(k) - \vec{q} \cdot \vec{k} - 2M_0^2 \right] \} . \end{aligned} \quad (38)$$

In this expression  $\Delta_i(k^2)$ ,  $i = \sigma, \omega$  is given by:

$$\Delta_i(k^2) = \frac{1}{q^2 - m_i^2 + i\epsilon}. \quad (39)$$

Note that the term proportional to  $k^\mu k^\nu / m_\omega^2$  in the vector-meson propagator is dropped due to the conservation of the baryon current.

Now we proceed by applying the PMS to  $\mathcal{E}_W^0$ :

$$\frac{d\mathcal{E}_W^{(0)}}{d\mu} = \frac{d\mathcal{E}_W^{(0)}}{dM_0} \frac{dM_0}{d\mu} = \frac{d\mathcal{E}_W^{(0)}}{dM_0} = 0. \quad (40)$$

At zeroth order in  $\delta$ , one can see from Eqs. (20) and (21) that no interaction between mesons and nucleons are considered. Thus,  $\Sigma^{(0)} = 0$ . On the other hand,  $M_0$  depends on  $\mu$ , vide Eq. (22) and it is precisely this parameter, fixed by the PMS condition, which introduces all the non perturbative information related to the interactions. Although we are working at zeroth order in  $\delta$ , contributions from direct and exchange terms are included in equations (35) - (38) above.

Let us first consider the contribution from the direct terms only, which are given by Eqs. (35) and (37). Application of the PMS to them yields the following self-consistency condition for  $M_0$ :

$$M_0 = M - \frac{g_\sigma^2}{m_\sigma^2} \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{M_0}{E_0(q)}. \quad (41)$$

This is exactly the same self-consistency condition for the effective nucleon mass obtained by means of the Hartree, or mean-field, approximation.

Now, application of the PMS to the full energy density leads to a nonlinear equation for  $\mu$ , or equivalently for  $M_0$ , which is more complicated than the one of Eq. (41). To avoid this cumbersome expression, we have chosen to find the minimum of the energy density

numerically. In Figure 1 we compare the nucleon binding energy,  $E/A - M$ , obtained by using only Eq. (34) and the direct contributions from Eqs. (35) and (37) (solid line) and coupling constants fixed by fitting the binding energy and density of equilibrium nuclear matter, with the full binding energy, keeping the same coupling constants (dotted line). The value of the coupling constants are  $g_s^2 = 91.64$  and  $g_v^2 = 136.2$ . The masses used in all calculations are  $M = 939$  MeV,  $m_v = 783$  MeV and  $m_\sigma = 550$  MeV. We find that the full result coincides with those obtained in a relativistic Hartree-Fock calculation [14,16] which we also show for comparison (long-dashed line). Note that the dotted and long-dashed lines coincide in the figure. Of course, one could renormalize the model parameters to reproduce the bulk saturation properties of nuclear matter. This would give us the same coupling constants used in the the relativistic Hartree-Fock calculation of Ref. [16]. Therefore, the PMS condition on the energy density obtained with the zeroth order propagator of the Walecka model is also equivalent to the usual Hartree-Fock solution. This is indeed a very interesting result since the self-energy expressions are not present and therefore only the exchange contributions to the energy density are enough to reproduce, through the minimization of this expression, the usual Hartree-Fock result.

In Figure 2 we compare the results for the effective nucleon mass in nuclear matter as a function of  $P_F$  obtained from  $\mu$ . From this figure, it is clear that the results with the exchange terms and renormalized constants coincide with the results obtained by using the direct terms only.

Next we check how the previous results change by dressing the nucleon propagator up to  $O(\delta^2)$ . For this purpose, we start from the calculation of the self-energy.

For infinite nuclear matter, because of the translational, rotational, parity and time reversal invariances,  $\Sigma(q)$  can be generally written in terms of the unit matrix and the Dirac  $\gamma_\mu$  matrices as follows [16]:

$$\begin{aligned}\Sigma(q) &= \Sigma^s(q) - \gamma_\mu \Sigma^\mu(q) \\ &= \Sigma^s(q^0, |\vec{q}|) - \gamma^0 \Sigma^0(q^0, |\vec{q}|) + \vec{\gamma} \cdot \vec{q} \Sigma^v(q^0, |\vec{q}|).\end{aligned}\quad (42)$$

The self-energy to second-order in delta is given by:

$$\Sigma^{(2)}(p) = -\mu\delta + i\frac{g_\sigma^2\delta^2}{m_\sigma^2} \int \frac{d^4q}{(2\pi)^4} \text{Tr} [S^{(0)}(q)] - i\frac{g_\omega^2\delta^2}{m_\omega^2} \int \frac{d^4q}{(2\pi)^4} \gamma_\mu \text{Tr} [\gamma^\mu S^{(0)}(q)]$$

$$+ ig_\sigma^2 \delta^2 \int \frac{d^4 q}{(2\pi)^4} S^{(0)}(q) \Delta_\sigma[(p-q)^2] - ig_\omega^2 \delta^2 \int \frac{d^4 q}{(2\pi)^4} \gamma_\mu S^{(0)}(q) \Delta_\omega[(p-q)^2] \gamma_\mu , \quad (43)$$

where the superscript (2) means second order in  $\delta$  and  $\Delta_\sigma$  and  $\Delta_\omega$  are given in Eq. (39). Again, we have made use of the baryon current conservation.

It is important to notice that this *is not* a self-consistent equation for  $\Sigma^{(2)}$ , although its formal similarity with the corresponding Hartree-Fock ones [14]. The r.h.s. of this equation is expressed in terms of functions calculated at the zeroth-order in  $\delta$ , as is usual in a perturbative calculation. We evaluate Eq. (43) neglecting the Feynman part of the nucleon propagator and considering just  $S^0(q)$  given by Eq. (29). Because of this, all integrals in Eq. (43) are finite and can be easily evaluated. The first term in Eq. (43) comes from the first order contribution in  $\delta$  and must be kept at second order. These expressions are very similar to the ones obtained with the Hartree-Fock approximation [16]. Since there are subtle differences, we write them explicitly below.

$$\begin{aligned} \Sigma^{s(2)}(p) = & -\delta\mu - \gamma \frac{g_\sigma^2 \delta^2}{m_\sigma^2} \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{M_0}{E_0(q)} \\ & + \frac{1}{4\pi^2 p} \int_0^{P_F} dq q \frac{M_0}{E_0(q)} \left[ \frac{1}{4} g_\sigma^2 \delta^2 \Theta_\sigma(p, q) - g_\omega^2 \delta^2 \Theta_\omega(p, q) \right] , \end{aligned} \quad (44)$$

$$\Sigma^{0(2)}(p) = -\gamma \frac{g_\omega^2 \delta^2}{m_\omega^2} \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} - \frac{1}{4\pi^2 p} \int_0^{P_F} dq q \left[ \frac{1}{4} g_\sigma^2 \delta^2 \Theta_\sigma(p, q) + \frac{1}{2} g_\omega^2 \delta^2 \Theta_\omega(p, q) \right] , \quad (45)$$

$$\Sigma^{v(2)}(p) = -\frac{1}{4\pi^2 p^2} \int_0^{P_F} dq q \frac{q}{E_0(q)} \left[ \frac{1}{2} g_\sigma^2 \delta^2 \Phi_\sigma(p, q) + g_\omega^2 \delta^2 \Phi_\omega(p, q) \right] , \quad (46)$$

where the functions  $\Theta_i(p, q)$ ,  $\Phi_i(p, q)$ ,  $i = \sigma, \omega$ , are defined by:

$$\Theta_i(p, q) = \ln \left| \frac{A_i(p, q) + 2pq}{A_i(p, q) - 2pq} \right| , \quad (47)$$

$$\Phi_i(p, q) = \frac{1}{4pq} A_i(p, q) \Theta_i(p, q) - 1 , \quad (48)$$

where

$$A_i(p, q) = \vec{p}^2 + \vec{q}^2 + m_i^2 - [E(p) - E_0(q)]^2 . \quad (49)$$

One should pay attention to the fact that now the self-energy also carries direct and exchange contributions.

We are in the position to calculate the energy density. We start by defining the following auxiliary quantities [16]:

$$\begin{aligned}
M^*(q) &\equiv M_0 + \Sigma^{s(2)}(q) , \\
\vec{q}^* &\equiv \vec{q} \left[ 1 + \Sigma^{v(2)}(q) \right] , \\
E^*(q) &\equiv \left[ \vec{q}^{*2} + M^{*2}(q) \right]^{\frac{1}{2}} , \\
q^{*\mu} &= q^\mu + \Sigma^{\mu(2)}(q) = \left[ q^0 + \Sigma^{0(2)}(q), \vec{q}^* \right] ,
\end{aligned} \tag{50}$$

and writing the nucleon propagator in the compact form:

$$S(q) = S_F(q) + S_D(q) , \tag{51}$$

$$S_F(q) = \left[ \gamma^\mu q_\mu^* + M^*(q) \right] \frac{1}{q^{*\mu} q_\mu^* - M^{*2}(q) + i\epsilon} , \tag{52}$$

$$S_D(q) = \left[ \gamma^\mu q_\mu^* + M^*(q) \right] \frac{i\pi}{E^*(q)} \delta(q^0 - E(q)) \theta(P_F - |\vec{q}|) , \tag{53}$$

where  $E(q)$  is the single-particle energy:

$$E(q) = \left[ E^*(q) - \Sigma^{0(2)}(q) \right] . \tag{54}$$

Note that we have assumed that the nucleon propagator has simple poles with unit residue. Within the approximation scheme we are working in this paper, this assumption is satisfied, as can be seen below.

At this order, for the functions  $\Delta_\sigma(k)$ ,  $\Delta_\omega^{\mu\nu}(k)$ ,  $\Pi_\sigma(k)$  and  $\Pi_\omega^{\mu\nu}(k)$  one has the same expressions as in Eqs. (30-33), where instead of  $S^0$  one uses the  $S$  above. In what follows, vacuum contributions have again been neglected. Hence, we obtain for the energy density of nuclear matter the following expression:

$$\mathcal{E}_W^{(2)} = \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{\vec{q} \cdot \vec{q}^* + M M^*(q)}{E^*(q)} + \mathcal{E}_\sigma^{(2),dir} + \mathcal{E}_\sigma^{(2),exc} + \mathcal{E}_\omega^{(2),dir} + \mathcal{E}_\omega^{(2),exc} , \tag{55}$$

with

$$\mathcal{E}_\sigma^{(2),dir} = -\frac{1}{2} \frac{g_\sigma^2}{m_\sigma^2} \left[ \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{M^*(q)}{E^*(q)} \right]^2 \tag{56}$$

$$\begin{aligned}
\mathcal{E}_\sigma^{(2),exc} &= \frac{g_\sigma^2}{2} \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3 E^*(q)} \int_0^{P_F} \frac{d^3 k}{(2\pi)^3 E^*(k)} \{ \Delta_\sigma[(q-k)^2] \\
&\times \left[ \left( \frac{1}{2} - 1 \right) - [E(q) - E(k)]^2 \Delta_\sigma[(q-k)^2] \right] \left[ q^{*\mu} k_\mu^* + M^*(q) M^*(k) \right] ,
\end{aligned} \tag{57}$$

$$\mathcal{E}_\omega^{(2),dir} = \frac{1}{2} \frac{g_\omega^2}{m_\omega^2} \left[ \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \right]^2 \tag{58}$$

$$\begin{aligned}
\mathcal{E}_\omega^{(2),exc} &= g_\omega^2 \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3 E^*(q)} \int_0^{P_F} \frac{d^3 k}{(2\pi)^3 E^*(k)} \{ \Delta_\omega[(q-k)^2] \\
&\times \left[ \left( \frac{1}{2} - 1 \right) - [E(q) - E(k)]^2 \Delta_\omega[(q-k)^2] \right] \left[ q^{*\mu} k_\mu^* - 2 M^*(q) M^*(k) \right] \} .
\end{aligned} \tag{59}$$

These expressions are very similar in form to the ones obtained in the Hartree-Fock approximation. However, one should notice that the self-energies are not calculated self-consistently as in the Hartree-Fock approximation, rather they are given by Eqs. (44) - (46), which depend on  $M_0$ , which by its turn, is determined numerically by minimizing the energy density. Also, differences are contained in the fermion kinetic energy, the first term in Eq. (55), and in the factors  $(\frac{1}{2} - 1)$  in Eqs. (56) and (58). These differences arise because we are not using the nucleon Schwinger-Dyson equation. Please refer to the Appendix for an explicit derivation in the case one chooses to eliminate the meson field operators [13] from the beginning. Application of the PMS to the direct contributions present in Eqs. (56) and (58), calculated only with the direct contributions to the self-energies, yields again the familiar Hartree result, i.e.,

$$M^* = M - \frac{g_\sigma^2}{m_\sigma^2} \gamma \int_0^{P_F} \frac{d^3 q}{(2\pi)^3} \frac{M^*}{E^*(q)} . \quad (60)$$

From this result it is straightforward to see that when only the direct terms are considered in the energy density and self-energies, the mean-field solution is reproduced at any order in  $\delta$ . This result should be compared with the one presented in Ref. [4] where, in the context of the effective potential, it was found that the  $\delta$ -expansion and the  $1/N$  expansion are identical in the large  $N$  limit.

For the full energy density, Eq. (55) has to be minimized in terms of  $\mu$  and this is done numerically. This is indeed simpler than the traditional Hartree-Fock procedure, where three coupled equations (the self-energy expressions) have to be solved self-consistently.

We do not present in Figure 1 the  $O(\delta^2)$  binding energy because it would be indistinguishable from the HF one. Instead, for comparison purposes, we present in Table 1 the results obtained with the self-consistent Hartree-Fock approximation and the ones with the  $\delta^0$  and  $\delta^2$  expansions. We note that a simple iterative procedure for solving the Hartree-Fock equations do not converge for Fermi momenta larger than  $P_F \sim 1.7$  fm $^{-1}$ . Inspection of the Table reveals the nice convergence towards the Hartree-Fock approximations of the results from  $\delta^0$  to  $\delta^2$ . Moreover, one sees that in order to reproduce the Hartree-Fock results, it is enough to use the simple calculation at zeroth order.

The behavior of  $M^*$  as a function of the Fermi momentum at this order does not show any noticeable difference as compared with the zeroth order results. In Figure 3 we plot

the energy density  $\mathcal{E}$  as a function of  $\mu$  for  $P_F = 1.19 \text{ fm}^{-1}$ . The solid line is obtained without the inclusion of the exchange term (the PMS solution in this case is given by  $\mu/M = -0.275$ ) and the dashed line gives the full second order density energy (the PMS solution is  $\mu/M = -0.35$ ). Recall that if one had an exact solution, the energy density would be independent of  $\mu$ . In this sense it is gratifying to notice that  $\mathcal{E}$  is a very flat function of  $\mu$ . This stability in the value of the energy density as a function of  $\mu$  is very desirable and guarantees that even big changes in the value of  $\mu$  will not affect physical quantities, as the binding energy for instance.

It is important to point out that although we have obtained the same results for the binding energy within the zeroth and second order approximations, this is not true at all orders when exchange terms are included. At fourth order in  $\delta$ , for example, vertex corrections will appear and the resulting energy density will certainly be different. In this work we have opted for neglecting vertex corrections in order to be able to compare our results with Hartree and Hartree-Fock results, where they are not included either. If vertex corrections are to be included, the full meson-nucleon vertex functions  $\Gamma_i$ ,  $i = \sigma, \omega$  appearing in Eqs. (16) and (17) and the full meson propagators will also have to be expanded in orders of  $\delta$ .

#### 4. Conclusions

In the third section of this paper we have utilized the optimized  $\delta$  expansion to study medium effects in the Walecka model. We have obtained results quantitatively similar to the ones of the usual Hartree-Fock approximation, although the analytical expressions are not evidently equivalent. If one neglects the exchange term in the energy density and self-energies then clearly the mean-field solution is reproduced at any order. This outcome reflects the fact that this perturbative method generates nonperturbative results due, of course, of the variational nature of the PMS.

Figure 1 and Table 1 show that the very simple calculation at zeroth order in  $\delta^0$  already provides a very good approximation to the Hartree-Fock results, at least for densities not much higher than the normal nuclear matter density. Analytically, this calculation is indeed simpler than the usual Hartree-Fock approximation, in view of the perturbative nature of the method. Numerically, this calculation is straightforward because no self-consistency has to be achieved; one needs only to perform a minimization of the energy with respect to the

parameter  $\mu$ . It is also worth mentioning that, in the Walecka model, the energy density is a very flat function of  $\mu$  and this guarantees that the PMS solution is indeed very stable.

On the basis of our results, we believe that the optimized  $\delta$  expansion is a very robust nonperturbative approximation scheme. Compared with the Hartree-Fock approximation, the  $\delta$  expansion is very economical because of its perturbative nature. Once the reliability of the scheme has been established, one is ready to proceed to other interesting applications. These include vertex and, obviously, vacuum effects that include exchange corrections [13]. In view of our results, we can proceed by including vertex corrections in the energy density and still maintaining the nucleon propagator at zeroth order in  $\delta$ . The study of the vacuum in the Walecka model is an important issue since one needs to know the limits of applicability of such model to high densities and/or temperatures before quark and gluon degrees of freedom have to be invoked. Particularly interesting is the renormalization of exchange diagrams which should become simplified in the present approach as compared with the Hartree-Fock scheme [17], since at each order in  $\delta$ , only a finite number of diagrams has to be taken into account.

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TABLES

TABLE I. Results for  $E/A - M$  (MeV) as a function of  $P_F$  (fm $^{-1}$ ) calculated with the Hartree-Fock ( $E_{HF}$ ) approximation and with the  $\delta^0$  ( $E_{\delta^0}$ ) and  $\delta^2$  ( $E_{\delta^2}$ ) expansions.

$P_F$	$E_{HF}$	$E_{\delta^0}$	$E_{\delta^2}$
0.05	0.0291095	0.0291095	0.0291095
0.25	0.5317530	0.5317529	0.5317530
0.50	1.1637572	1.1637562	1.1637569
0.75	0.5920830	0.5920533	0.5920831
1.00	-1.8171553	-1.8175904	-1.8171548
1.25	-4.0899150	-4.0937442	-4.0898972
1.50	5.7406683	5.7109528	5.7409897
1.65	31.0297464	30.9322795	31.0312656

## FIGURES

FIG. 1.  $P_F$  dependence of the binding energy of the Walecka model at zeroth order in  $\delta$ . The solid line represents the direct contributions only (Eqs.(34),(35),(37)). The dotted and long-dashed lines give the full binding energy and the Hartree-Fock solution respectively, both determined with the same coupling constants used in the solid line solution (note that the lines are coincident).

FIG. 2. Zeroth order nucleon effective mass  $M_0$  as a function of  $P_F$ . The solid curve is the result obtained without the exchange term and the dashed curve is the result using the full energy density.

FIG. 3.  $\mu$  dependence of the energy density for the Walecka model at second order in  $\delta$ , calculated at  $P_F = 1.19 \text{ fm}^{-1}$ . The solid line gives the solution when the exchange term is not included. The dashed line gives the full solution.

## Appendix

For completeness, in this Appendix we employ the method used in Ref. [13] for obtaining the general expression for the energy density in terms of the nucleon propagator, valid up to order  $\mathcal{O}(\delta^2)$ .

The energy-momentum tensor is defined by Eq. (12) with  $\mathcal{L}_W$  defined in Eq. (14):

$$T_W^{\mu\nu} = T_B^{\mu\nu} + T_\sigma^{\mu\nu} + T_\omega^{\mu\nu} \quad (61)$$

with

$$T_B^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma_\alpha\partial^\alpha - M)\psi, \quad (62)$$

$$T_\sigma^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\left(g_\sigma\bar{\psi}\phi\psi - \frac{1}{2}m_\sigma^2\phi^2 + \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi\right), \quad (63)$$

$$T_\omega^{\mu\nu} = \partial^\nu V_\lambda F^{\lambda\mu} - g^{\mu\nu}\left(-g_\omega\gamma_\alpha V^\alpha + \frac{1}{2}m_\omega^2 V_\alpha V^\alpha - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right). \quad (64)$$

Let us concentrate on  $T_\sigma^{\mu\nu}$ . The Euler-Lagrange equation for the scalar-meson field equation is:

$$\left(\partial_\mu\partial^\mu + m_\sigma^2\right)\phi = g_\sigma\bar{\psi}\psi. \quad (65)$$

This equation can formally be integrated as:

$$\phi(x) = \phi^0(x) - g_\sigma \int d^4y \Delta_\sigma(x-y) \bar{\psi}(y)\psi(y), \quad (66)$$

where  $\phi^0$  is the solution of the homogeneous equation and  $\Delta_\sigma(x)$  is given by:

$$\Delta_\sigma(x) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_\sigma^2 + i\epsilon} e^{-iqx} = \int \frac{d^4q}{(2\pi)^4} \Delta_\sigma(q^2) e^{-iqx}. \quad (67)$$

From Eq. (66) (note that the first term  $\phi^0(x)$  does not contribute) we have:

$$g_\sigma \langle \bar{\psi}\phi\psi \rangle = -g_\sigma^2 \int d^4y \Delta_\sigma(x-y) \langle \bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)\psi_\beta(y)\psi_\alpha(x) \rangle. \quad (68)$$

With the help of Wick's contraction technique we get

$$\begin{aligned} g_\sigma \langle \bar{\psi}\phi\psi \rangle &= g_\sigma^2 \int d^4y \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \Delta_\sigma(p^2) \\ &\quad \times \left[ \int \frac{d^4q}{(2\pi)^4} \text{Tr}[S(q)] \int \frac{d^4k}{(2\pi)^4} \text{Tr}[S(k)] \right. \\ &\quad \left. - \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \int \frac{d^4k}{(2\pi)^4} e^{-ik(y-x)} \text{Tr}[S(q)S(k)] \right], \end{aligned} \quad (69)$$

that can finally be written as

$$g_\sigma \langle \bar{\psi} \phi \psi \rangle = -\frac{g_\sigma^2}{m_\sigma^2} \left[ \int \frac{d^4 q}{(2\pi)^4} \text{Tr}[S(q)] \right]^2 - g_\sigma^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \text{Tr}[S(q+k)S(k)] \Delta_\sigma(q^2). \quad (70)$$

The third term in Eq. (63) above can be written as

$$-\frac{1}{2}m_\sigma^2 \langle \phi^2 \rangle = -\frac{1}{2}m_\sigma^2 g_\sigma^2 \int d^4 y d^4 z \Delta_\sigma(x-y) \Delta_\sigma(x-z) \langle \bar{\psi}_\alpha(y) \psi_\alpha(y) \bar{\psi}_\beta(z) \psi_\beta(z) \rangle. \quad (71)$$

Using again Wick's technique

$$\begin{aligned} -\frac{1}{2}m_\sigma^2 \langle \phi^2 \rangle &= \frac{1}{2}m_\sigma^2 g_\sigma^2 \int d^4 y d^4 z \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} e^{-ip(x-y)} \Delta_\sigma(p^2) e^{-iq(x-z)} \Delta_\sigma(q^2) \\ &\quad \times \left[ \int \frac{d^4 k}{(2\pi)^4} \text{Tr}[S(k)] \int \frac{d^4 k'}{(2\pi)^4} \text{Tr}[S(k')] \right. \\ &\quad \left. - \int \frac{d^4 k}{(2\pi)^4} e^{-ik(z-y)} \int \frac{d^4 k'}{(2\pi)^4} e^{-ik'(y-z)} \text{Tr}[S(k)S(k')] \right], \end{aligned} \quad (72)$$

and finally

$$\begin{aligned} -\frac{1}{2}m_\sigma^2 \langle \phi^2 \rangle &= \frac{1}{2} \frac{g_\sigma^2}{m_\sigma^2} \left[ \int \frac{d^4 q}{(2\pi)^4} \text{Tr}S(q) \right]^2 \\ &\quad - \frac{1}{2}m_\sigma^2 g_\sigma^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \Delta_\sigma(q^2) \text{Tr}[S(q+k)S(k)] \Delta_\sigma(q^2). \end{aligned} \quad (73)$$

Following the same procedure for  $\partial^0 \phi \partial^0 \phi$  and  $\partial_\mu \phi \partial^\mu \phi$ , one obtains after adding all terms:

$$\begin{aligned} T_s^{00} &= +\frac{1}{2} \frac{g_\sigma^2}{m_\sigma^2} \left[ \int \frac{d^4 q}{(2\pi)^4} \text{Tr}S(q) \right]^2 + g_\sigma^2 \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \text{Tr}[S(q+k)S(k)] \Delta_\sigma(q^2) \\ &\quad \times \left[ (q^0)^2 \Delta_\sigma(q^2) + \frac{1}{2}(q^2 + m_\sigma^2) \Delta_\sigma(q^2) \right]. \end{aligned} \quad (74)$$

The same can be repeated for the vector-meson field. This shows the equivalence up to  $\mathcal{O}(\delta^2)$  of the perturbative solution of the Schwinger-Dyson equations and the method of elimination of the meson field equations from the beginning [13].





